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ON THE THEORY OF AXISYMMETRIC FLOW OF A SUPERSONIC STREAM OF GAS PAST A POINTED BODY OF REVOLUTION

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The axisymmetric stationary flow of a supersonic stream of nonviscous gas past a pointed body of revolution, the curvature of whose meridional curve is nonzero at the vertex of the body, is considered. It is shown that close to the body surface the derivatives of the specific entropy $S = S(s, \zeta)$ and the tangent velocity component of gas particles $u = u(s, \zeta)$ with respect to $\zeta = n / s$ (where n is the distance from the vertex of the body along the meridional curve and s is the distance along the normal to the surface of the body) are of the order of $\zeta^{-1/2}$, i.e. the body surface is singular.

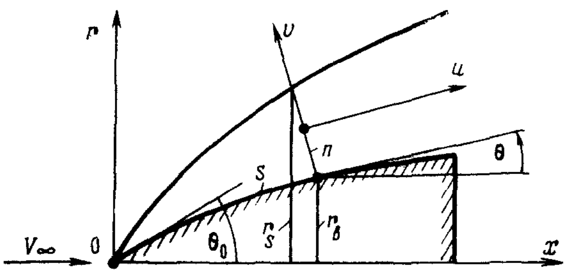

Fig. 1

1. Let us consider the axisymmetric steady flow at Mach number $M_\infty > 1$, velocity V_∞, etc. (see Fig. 1), past a pointed body of revolution of a supersonic stream of nonviscous gas. We introduce cylindrical coordinates x and r with the axis Ox that coincide with the axis of symmetry of the body, and the coordinates s and n used in the theory of boundary layers. The streamlined body surface is defined by the equation $r = r_b(s)$, $dr_b / ds = \sin\theta$, where θ is the angle between the tangent to the meridian curve and the axis Ox and $d\theta / ds = -\varkappa$, where $\varkappa$ is the curvature of the meridian curve. Euler's equations, of continuity and energy for steady flows arising at the intersection of a homogeneous stream of shock waves, can be transformed with the use of Bernoulli's integral to a form in which the velocity $\mathbf{V}$ of gas particles and the specific entropy S are the unknown functions. The analogous conversion for conical streams of gas is made in [1].

For the axisymmetric problem, where s and $\zeta = n / s$ are taken as independent variables, the transformed equations are of the form

$$\Big\{(u^2 - a^2)\left(s\frac{\partial u}{\partial s} - \zeta\frac{\partial u}{\partial \zeta}\right) + (v^2 - a^2)(1 + \varkappa\zeta s)\frac{\partial v}{\partial \zeta} + uv\left[s\frac{\partial v}{\partial s} - \zeta\frac{\partial v}{\partial \zeta} + (1 + \varkappa\zeta s)\frac{\partial u}{\partial \zeta}\right]\Big\}\frac{r_b + \zeta s\cos\theta}{s} - a^2[(1 + \varkappa\zeta s)(u\sin\theta + v\cos\theta) + (r_b + \zeta s\cos\theta)\varkappa v] = 0 \tag{1.1}$$

$$u\left[s\frac{\partial v}{\partial s}-\zeta\frac{\partial v}{\partial \zeta}-(1+\varkappa\zeta s)\frac{\partial u}{\partial \zeta}\right]-\varkappa u^2 s-T(1+\varkappa\zeta s)\frac{\partial S}{\partial \zeta}=0 \qquad (1.2)$$

$$[v(1+\varkappa\zeta s)-\zeta u]\frac{\partial S}{\partial \zeta}+us\frac{\partial S}{\partial s}=0 \qquad (1.3)$$

Here u and v are components of $\mathbf{V}$ in the direction of increase of s and n, respectively, $T=T(S,i)$ is the absolute temperature, and $a=a(S,i)$ is the speed of sound which are given functions of their own arguments, where $i=i_\infty+{}^1/_2(V^2-V_\infty^2)$ is the specific enthalpy. The condition of separation-free flow over the surface of the body is of the form $v=0$, $\xi=0$. At the bow shock wave, defined by the equation $r_b=r_b(s)$ (see Fig. 1), conditions of compatibility which are not adduced here, must be satisfied. The stream flowing past the body is assumed to be supersonic, therefore close to the vertex of the body the flow is almost conical, i.e. when $s\to 0$

$$U(s,\zeta)\to u_1(\zeta),\quad v(s,\zeta)\to v_1(\zeta),\quad S(s,\zeta)\to S_1=\text{const}$$

where u_1, v_1, S_1 are parameters of conical flow around a cone with the vertex half-angle θ_0 (see Fig. 1).

2. Let us consider Eq. (1.3). We substitute the variable $\xi=\zeta^{1/2}$ for ζ, then (1.3) may be written in the form

$$\frac{v(1+\varkappa\zeta s)-\zeta u}{2\zeta}\frac{\partial S}{\partial \xi}+us\frac{\partial}{\partial s}\left(\frac{S-S_b}{\xi}\right)=0 \qquad (2.1)$$

where $S_b=\text{const}$ is the value of entropy at the surface of the body. Passing in (2.1) to the limit with ξ tending to zero, we obtain

$$\frac{1}{2u}\left(\frac{\partial v}{\partial \zeta}-u\right)\frac{\partial S}{\partial \xi}+s\frac{d}{ds}\left(\frac{\partial S}{\partial \xi}\right)=0 \qquad (2.2)$$

Here all functions are taken for $\xi=0$, i.e. on the body surface. Solving Eq. (2.2) for $(\partial S/\partial\xi)_{\xi=0}$, we obtain

$$\left(\frac{\partial S}{\partial \xi}\right)_{\xi=0}=c\cdot\exp\left[\int_{s_0}^{s}\frac{1}{2u}\left(u-\frac{\partial v}{\partial \zeta}\right)\frac{ds}{s}\right] \qquad (2.3)$$

where c and s_0 are constants.

We now find $\partial v/\partial\zeta$ for $\xi=0$ from Eq. (1.1). Passing to the limit at $\zeta\to 0$, we obtain from (1.1) the relation

$$\frac{r_b}{s}\left[(u^2-a^2)s\frac{\partial u}{\partial s}-a^2\frac{\partial v}{\partial \zeta}\right]-a^2\sin\theta u=0$$

from which follows

$$\partial v/\partial\zeta\to -u\quad \text{for } s\to 0\quad (\xi=\zeta=0) \qquad (2.4)$$

From this and from formula (2.3) we find that

$$\left(\frac{\partial S}{\partial \xi}\right)_{\xi=0}\sim s\quad \text{for } r\to 0,\quad \left(\frac{\partial S}{\partial \xi}\right)_{\xi=0}\equiv 0$$

If it can be shown that $(\partial^2 S/\partial s\partial\xi)_{s=\xi=0}\neq 0$, then this will mean that the constant c in formula (2.3) is nonzero, and that in the vicinity of the body surface there is a layer where

$$S-S_b\sim\xi\sim\zeta^{1/2}\sim(n/s)^{1/2} \qquad (2.5)$$

To show that $(\partial^2 S/\partial s\partial\xi)_{s=\xi=0}\neq 0$, we establish a system of equations for the functions

$$u_2(\zeta)=\lim_{s\to 0}\frac{\partial u}{\partial s},\quad v_2(\zeta)=\lim_{s\to 0}\frac{\partial v}{\partial s},\quad S_2(\zeta)=\lim_{s\to 0}\frac{\partial S}{\partial s}$$

(Finding u_2, v_2, S_2 is of interest in itself, since these functions represent the effect of the quantity $\varkappa_0 = \varkappa|_{s=0}$ on the field of flow). To determine u_2, v_2, S_2, it is necessary to solve the linear boundary value problem, in which the condition of flow around the body and at the bow shock wave must be satisfied. For $s = 0$ the "free parameter" of the problem is the curvature of the bow shock wave meridian curve whose choice must be such that the condition $v_2 = 0$ and $\zeta = 0$ of flow around the body is satisfied. The expression for S_2 has the form

$$S_2(\zeta) = S_{2s} \exp\left(\int_{\xi_0}^{\zeta} \frac{u_1}{\xi u_1 - v_1} d\zeta\right) \tag{2.6}$$

where $S_{2s} = \text{const}$; u_1, v_1 are parameters of conical flow around a cone with the vertex half-angle θ_0; $\xi = \xi_0$ is the equation of a cone touching the shock wave at the vertex of the body. Calculations showed that $S_{2s} \neq 0$, in particular in the case of a perfect gas:

$$S_{2s} = -{}^3/_2\, \varkappa_0 \operatorname{ctg} \theta_0\, c_p \quad \text{for } M_\infty \to \infty,\ \gamma \to 1$$

where c_p is the specific heat of the gas at constant volume, γ is the adiabatic exponent. Since $dv_1/d\zeta \to -u_1$ for $\xi \to 0$ (see (2.4)), it follows from formula (2.6) that

$$S_2(\zeta) = \lim_{s\to 0} \frac{\partial S}{\partial s} \sim \zeta^{1/2} \sim \xi, \quad \xi \to 0$$

Differentiating this equation with respect to ξ, we obtain

$$\left(\frac{dS_2}{d\xi}\right)_{\xi=0} = \left(\frac{\partial^2 S}{\partial\xi\partial s}\right)_{\xi=s=0} \neq 0$$

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From Eq. (1.2), for $\xi \to 0$ follows that the term $T(1 + \varkappa \zeta s)\, \partial S/\partial\zeta$ of order $\zeta^{-1/2}$, may be compensated for only by the term c'' of the derivative $\partial u/\partial\zeta$. This means that $\partial u/\partial\zeta \sim \xi^{-1/2}$ when $\xi \to 0$.

Thus, the following result is obtained.

If there exists a unique solution of the problem of an axisymmetric supersonic flow past a pointed body of revolution whose second derivatives with respect to s and $\zeta = n/s$ are continuous in a certain area around the vertex of the body, then in the case of $\varkappa_0 \neq 0$ the surface of the streamlined body is singular, in its vicinity $\partial u/\partial\zeta$, $\partial S/\partial\zeta \sim \zeta^{-1/2}$ or $\partial u/\partial n$, $\partial S/\partial n \sim n^{-1/2}$ when $n \to 0$ and s has a fixed value.

REFERENCE

1. Bulakh, B. M., Nonlinear conical flows of gas. Moscow, "Nauka", 1970.

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